

## LETTERS

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### Gravity flow of a viscous liquid down a slope with injection

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The free-surface lubrication equations are solved numerically for the time-dependent three-dimensional flow down an inclined plane produced by continuous injection through a circular orifice in the plane. The flow pattern ultimately becomes steady with stream depth and width scaling as the  $-\frac{1}{3}$ th power and  $\frac{2}{3}$ th power of the downslope distance from the orifice, in agreement with a known similarity solution. The predicted stream width gives partial agreement with a recent experimental result, even though significant thermal effects were present in the experiment. The physical prototype is flow of lava.

Recently we have developed a numerical model for gravity-driven flows over complex topography, including thermal effects and possible non-Newtonian behavior when the fluid is a two-phase mixture. The envisioned applications come primarily from geophysics and include flow of lava, mudslides, and related phenomena. In most of these cases the "coating" layer is thin, relative to characteristic horizontal dimensions, and is, in addition, of slowly varying thickness; thus, for laminar flows at least, the quasiunidirectional or "lubrication" approximation is justifiable. Very substantial simplifications follow from this assumption, including the reduction from three space dimensions to two. Relatively straightforward numerical schemes for the time-dependent problem prove to be adequate, allowing incorporation of a variety of additional physical effects, such as heat loss, solidification, and temperature-dependent fluid properties. Attempts to understand the flow of lava, in particular, have a long history, ranging from qualitative descriptions in ancient times<sup>1</sup> to much more recent modeling studies.<sup>2-4</sup>

Imbedded in the family of complex problems is a rather classical example: the unsteady spreading flow of a fluid of constant viscosity down an inclined plane, the flow resulting from a constant rate of injection through a circular hole near the top of the plane. The governing equations, an indication of the numerical procedure, and the results of a model calculation are given below.

The model calculation indicates that the flow approaches a steady-state pattern when the injection is continued for a long time. At sufficient distances from the hole, or "vent," details of the injection should not be important, suggesting that the problem will approach the similarity solution given by Smith.<sup>5</sup> Indeed, we demonstrate close agreement with the long-time numerical result for distances as

small as 1 or 2 diam downhill from the vent.

Finally, we compare the theoretical results with one case studied in the recent laboratory experiments of Hallworth *et al.*<sup>6</sup> They use heated polyethylene glycol (PEG) as a surrogate for basaltic lava. The "lava" is injected at a constant rate down an inclined plane and, as it flows, loses heat and ultimately freezes. Even though the thermal effects are extremely important, the profile of the frozen stream for a distance of at least 12 diam downhill from the vent appears to follow the predicted  $\frac{2}{3}$ th power scaling law.

We choose a Cartesian coordinate system with the  $x$  and  $y$  axes lying on an inclined plane. The plane is tilted at an angle  $\alpha$  to the horizontal with the  $x$  axis pointing in the downhill direction. The height of the liquid layer  $z = h(x, y, t)$  is measured normal to the inclined plane. Conservation of mass and momentum, the latter in the context of fully developed, slowly varying laminar flow, i.e., the lubrication approximation, then yields

$$\mathbf{Q} = -(\rho g h^3 / 3\mu)(\cos \alpha \nabla h - \mathbf{i} \sin \alpha) \quad (1a)$$

and

$$\frac{\partial h}{\partial t} = -\nabla \cdot \mathbf{Q} + w_i(x, y), \quad (1b)$$

where the gradient and divergence are two-dimensional operators in  $x$  and  $y$ . The flow rate is defined by

$$\mathbf{Q} = \int_0^{h(x, y)} \mathbf{V}(x, y, z) dz = (Q_x, Q_y),$$

where  $\mathbf{V}$  is essentially parallel to the  $(x, y)$  plane. Here,  $\mathbf{i}$  is a unit vector in the  $x$  direction,  $\rho$  and  $\mu$  are the effective density and viscosity, respectively, and  $g$  is the acceleration of gravity. For constant  $h$ , the second term on the right in (1a) yields the flow rate for steady two-dimensional unidirectional flow

on an inclined plane,<sup>7</sup> while the first term is the additional flow produced by differences in layer thickness. Here,  $w_i$  is the injection velocity normal to the inclined plane. The liquid is assumed to be injected through a circle of radius  $R$ , centered about the point  $(x_0, 0)$ , with a parabolic velocity profile. Thus

$$w_i = [2\Gamma/(\pi R^2)][1 - (r^2/R^2)] \quad (2)$$

for  $r = [(x - x_0)^2 + y^2]^{1/2} < R$  and equal to zero otherwise. Here,  $\Gamma$  is the constant rate of volumetric injection.

Equations (1) and (2) may be made dimensionless using  $R$  as characteristic length and

$$\tau = 3\mu/(\rho g R) \quad (3)$$

as characteristic time. Then the only dimensionless parameters appearing in the problem are  $\alpha$  and the injection rate

$$\chi = 3\mu\Gamma/(\rho g R^4). \quad (4)$$

Numerical solutions have been produced by a straightforward finite difference method. The equations, written in conservation form, are discretized using central differences in space and either explicit or alternating-direction-implicit time marching. Time steps are selected small enough to ensure stability while accuracy is verified by convergence under mesh refinement. Results of a model calculation are shown in Fig. 1. The parameter values correspond to run B of the experimental paper by Hallworth *et al.*<sup>6</sup> This experiment utilized an inclined plane set at  $\alpha = 15^\circ$  and a flow rate  $\Gamma$  of  $6.0 \text{ cm}^3/\text{sec}$  through a vent of radius  $0.35 \text{ cm}$ . Their initial effective value of  $\mu/\rho$  was  $0.69 \text{ cm}^2/\text{sec}$ , which is held constant in our simulation. The resulting value of the parameter  $\chi$  is  $0.844$ . The mesh size used for the calculation in Fig. 1 is  $\Delta x = \Delta y = 0.4$  and the time step is taken as  $0.05$ . The mound shape is bilaterally symmetric about the line of maximum fall; thus only one-half of the field needs to be calculated. The overall mesh size is  $270 \times 50$ . The evolution of the mound was calculated from the start of injection until  $t = 1200$ , the entire calculation requiring about 2 h on a

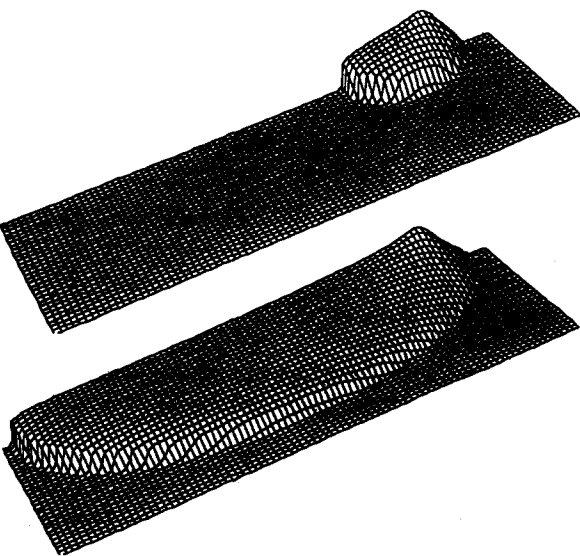


FIG. 1. Numerical simulation of flow down an inclined plane for  $\chi = 0.844$ . Only the right-half of bilaterally symmetric flow is shown and the vertical scale has been exaggerated by a factor of 4. Times are  $t = 37.5$  (top) and  $300$  (bottom).

SUN 3/160 Workstation using an optimized FORTRAN code and a floating point accelerator.

Of interest are the steep gradients near the downhill leading edge of the flowing mound in Fig. 1. Analytical studies of spreading mounds on a horizontal surface for both two-dimensional and axisymmetric geometries<sup>8</sup> have shown that  $h \sim (x_n - x)^{1/3}$  near the leading edge,  $x = x_n(t)$ . Extending the local analysis to the present case of an inclined plane, and assuming the flow to be two dimensional very near the moving front, we find that the first term on the right in Eq. (1a) is dominant and yields the same  $-1/3$  power behavior locally. Aside from a small "leakage" to each immediately adjacent downstream cell, the simulation faithfully reproduces the local behavior. Actually, the immediate neighborhood of a moving front is a region of nonuniformity for several reasons; these include the failure of the lubrication hypothesis when the slope is large, and, depending on the value of Bond number, neglect of surface tension and moving-contact-line/wetting phenomena. Physically, the strongly convective nature of the governing equation away from the neighborhood of the moving front, which is responsible for the steepness of the front, prevents the nonuniformity from contaminating other portions of the flow, though this observation would benefit from a more rigorous analysis. It is interesting to note, on the basis of the similarity solution discussed below, that the nonuniformity ultimately disappears on the sides of the mound since the mound meets the plane at uniformly small angles there.

It may be noted, by comparing the two parts of Fig. 1, that the shape of the surface in the vicinity of the vent remains virtually unchanged as time evolves. This suggests that the present flow will eventually reach a steady state, in contrast with spreading on horizontal surfaces<sup>8</sup> where the height of the mound increases without limit. Moreover, for regions downhill of the vent, details of the injection, including the characteristic dimension  $R$ , become unimportant. Thus a similarity solution involving only the parameters  $\chi$  and  $\alpha$  is anticipated.

Because of the spreading in the  $y$  direction, the liquid layer becomes thinner as  $x$  increases. Eventually, for sufficient distances downhill,

$$\left| \frac{\partial h}{\partial x} \right| \ll \tan \alpha, \quad (5)$$

consistent with the slow-variation assumption. Thus the downhill component of the first term on the right of Eq. (1a) may be dropped and the equation for the steady shape of the liquid becomes

$$\frac{\partial}{\partial y} \left( h^3 \frac{\partial h}{\partial y} \right) = \tan \alpha \frac{\partial h^3}{\partial x}. \quad (6)$$

Smith<sup>5</sup> has derived a similarity solution for Eq. (6). Using our notation and nondimensionalization, his result can be shown to be equivalent to

$$h(x, y) = \frac{0.9090}{x^{1/7}} \left( \frac{\chi^2}{\sin 2\alpha} \right)^{1/7} \times \left[ 1 - 0.2357 \tan \alpha \left( \frac{\sin 2\alpha}{\chi^2} \right)^{1/7} \frac{y^2}{x^{6/7}} \right]. \quad (7)$$

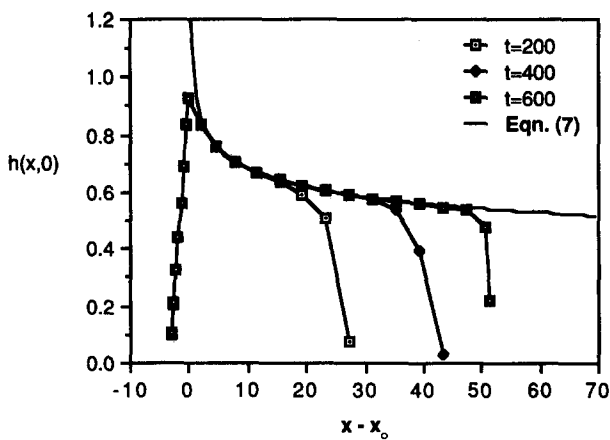


FIG. 2. Centerline height of the liquid mound at various times; comparison with steady-state similarity solution.

Experiments reported by Smith, using silicone oil, confirm the major features of the similarity solution.

In Figs. 2 and 3 we compare the similarity solution with the time-dependent numerical results. The distance  $x$  in the similarity solution, in effect an “outer” solution, is measured from the center of the vent; a more explicit result could be obtained by matching to a steady-state solution valid near the vent. Figure 2 compares the steady-state centerline height variation with points from the numerical solution for  $t = 200, 400,$  and  $600$ . Figure 3 shows a section through the mound corresponding to  $x = 15.2$ . The asymptotic profile is a parabola, from Eq. (7). For  $t = 1200$ , the last numerical result shown, good agreement with the similarity solution is apparent except close to the leading edge, where the small values of height are responsible for the slow approach to the asymptote.

The runs made in the experiment of Hallworth *et al.*<sup>6</sup> were designed to explore the effect of cooling on flowing lava, involving increased viscosity and ultimate solidification. The PEG flowed down the slope under water; the small solubility of PEG in water eliminated the effect of surface tension and the undesired length scale resulting from it. While the theory and computations presented here refer to flow at constant viscosity, some comparison is possible near the vent while the “lava” is still hot and its viscosity has not yet increased appreciably.

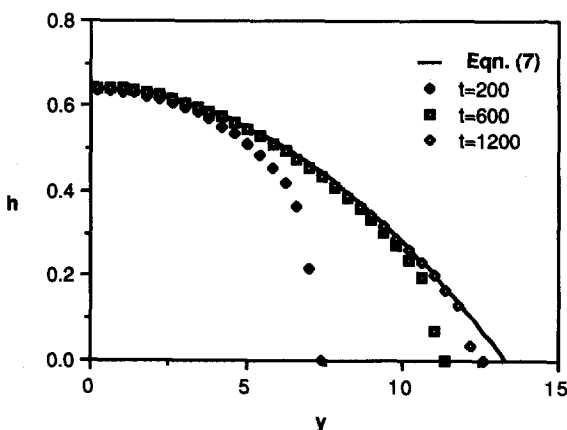


FIG. 3. Section through liquid mound for  $x - x_0 = 15.2$  at various times; comparison with steady-state similarity solution.

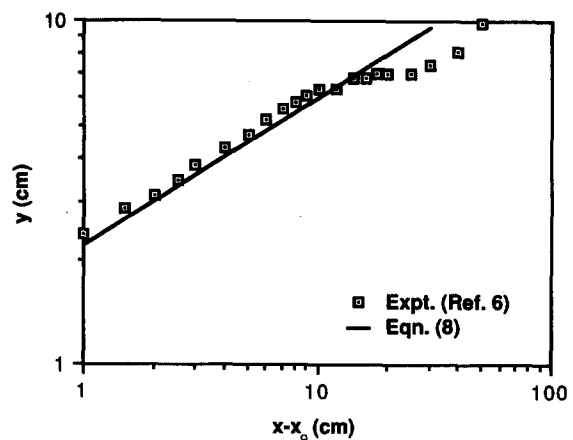


FIG. 4. Comparison of similarity solution with run B of Hallworth *et al.*<sup>6</sup>; width of steady-state mound  $y$  versus downhill distance  $x - x_0$ , both measured in centimeters.

A comparison of the similarity solution for stream width  $y(x)$ , using  $h = 0$  in Eq. (7) and the experimental parameter values, with the frozen profile from the experiment is shown in Fig. 4. Some irregularity was seen in the experimental picture and the flow was not quite bilaterally symmetric. The data shown for the experimental profile is the average value of  $y$ , using the right and left edges measured from an enlargement, at intervals of 1 cm. With  $y$  and  $x$  measured in centimeters, the equation of the line in Fig. 4 is

$$y = 2.24(x - x_0)^{3/7}. \quad (8)$$

For distances from the vent, between 1 and 10 cm, the power law behavior appears to be confirmed. Here,  $x_0$  is taken as the center of the vent. (The actual effective origin should be found by use of an “inner” solution. It obviously should be farther uphill, which would result, presumably, in somewhat closer agreement with the data.) At farther distances from the vent, solidification occurs in the experiment which is not accounted for in the analytical or the numerical results presented here.

Hallworth *et al.*<sup>6</sup> also present data for runs made at much lower rates of injection. While the trend is qualitatively as predicted by Eq. (7), i.e., narrower streams for lower values of  $\chi$ , solidification begins close to the vent and the streams are very much more irregular.

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