LETTERS

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Viscous flows down an inclined plane: Instability and finger formation

Leonard W. Schwartz
Departments of Mechanical Engineering and Mathematical Sciences, University of Delaware, Newark, Delaware 19716

(Received 18 October 1988; accepted 20 December 1988)

The free-surface lubrication equations, including the effects of both gravity and surface tension, are solved numerically for the evolution of thin liquid films on an inclined plane. The advancing contact angle is taken to be zero, equivalent to the assumption of perfect wetting. The shape of a flowing drop compares well, in detail, with a recently published interferometric study. A mound of liquid whose cross section is intially invariant in the horizontal direction is allowed to flow down a vertical plane with no slip imposed on the confining side walls. Fingers, or rivulets, form that are similar to those observed experimentally. The instability that leads to fingering is lateral flow driven by surface tension with larger curvature at the finger troughs. A scaling law for the prediction of finger width is derived.

Recently, we have published[1] a numerical study of flow down an inclined plane with continuous injection from a small hole near the top of the plane. This unsteady flow, driven by gravity, is a simplified model of the flow of lava out of the vent of a volcano, for example. The basis of the mathematical model is the lubrication assumption, i.e., that the product of Reynolds number based on fluid layer depth and the slope of the fluid layer, measured normal to the plane, is a small quantity. Such flows are also of interest at much smaller length scales; these "coating" flows are clearly relevant to painting and also to situations where thin liquid layers are used to maximize heat or mass transfer across the interface between a liquid and its vapor.

According to lubrication theory, the evolution of a thin liquid layer, flowing down a plane slope inclined at an angle α to the horizontal, is

$$h_t = \nabla\cdot[h^3(\cos\alpha\nabla h - B^{-1}\nabla\nabla^2 h - \mathbf{i}\sin\alpha)], \qquad (1)$$

where h is the layer thickness measured normal to the plane and ∇ is the two-dimensional Cartesian operator $(\partial/\partial x, \partial/\partial y)$. The x and y axes lie in the plane with the x-direction unit vector $\mathbf{i}$ pointing downhill. With L as the characteristic length for x, y, and h, the reciprocal Bond number is defined as

$$B^{-1} = \sigma/(\rho g L^2), \qquad (2)$$

where σ is surface tension, ρ is liquid density, and g is the acceleration of gravity. Consistent with lubrication theory, the mean curvature of the surface is approximated by $\nabla^2 h$. The time t is measured in units of $3\mu/(\rho g L)$. The first and third terms on the right side of Eq. (1) are as used previously by us[1]; here we also include the surface tension term.

We consider the collapse of a mound of liquid, and the prescription of $h(x,y)$ at $t=0$ is also required. The two cases treated here are an initially paraboloidal mound, corresponding to a sessile liquid droplet, flowing down a vertical wall, and a parabolic mound whose one-dimensional downhill flow is disturbed either by imposing a small-amplitude sinusoidal ripple or by imposition of a type of no-slip condition on the side walls of a finite-width plate. This last case may be compared with the experimental results of Huppert[2] or Silvi and Dussan V[3] for those cases where the liquids strongly wet the inclined plate. In the model, the advancing contact angle is taken to be zero, in the sense that no restriction is placed on the mass flow in the neighborhood of the apparent contact line. That is, the algorithm is applied uniformly over the mesh and the mobility h^3 of the empty cells is equal to zero.

The numerical procedure involves the discretization of the (x,y) plane into rectangular cells with the depth of the fluid layer h evaluated at the cell midpoints. Fluxes between cells are calculated at the cell boundaries; thus fluid volume is strictly conserved. Spatial derivatives are approximated using low-order central differences. Time integration is done explicitly with time steps taken sufficiently small to ensure stability. Accuracy is established by convergence under refinement. Typical computations employed a 60×180 square mesh and required several hours on either a VAX 785 or SUN 3/160 workstation with a floating point accelerator.

In Fig. 1 we show results for the flow of an initially paraboloidal drop that is allowed to flow down a vertical wall. The initial height is 0.25 mm and the radius, taken as the characteristic length in (2), is 5 mm; thus the droplet volume is about 10 $\mu\ell$. With σ taken as 20 dyn/cm and specific gravity of the liquid as 0.98 (e.g., silicone oil), the pa-

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rameter B^{-1} is equal to 0.083. Assuming a viscosity of 100 cP, the profile shown corresponds to a time t (after the start of the flow) of 150 sec. The height difference between contour lines in the figure is 8 μm. Also shown in Fig. 1 is a perspective drawing of one-half of the bilaterally symmetric droplet. The contour plot may be compared with Fig. 3 of Tanner,[4] who shows an interferogram taken from an experiment using silicone oil. In all qualitative respects agreement is seen. As he reports, the base of the droplet consists of two circular arcs connected by straight sides. A secondary peak is seen to develop above the primary one with a col, or saddle point, between them. If a section is taken about halfway down the drop, the centerline is a local minimum with the maximum about one-third of the way to the drop's sides; this feature is also present in the interferogram, although it is partially obscured by the presence of what appears to be a contaminant particle. Specific values of physical parameters for the experiment are not given, but the contact angle is certainly small.

The instability of a falling liquid film is commonly observed, e.g., when washing a windshield or painting a vertical wall, where insufficient care leads to the formation of "drip marks." Only recently, however, have controlled experiments been performed[2,3] to quantify the phenomenon. As in the experiments, we consider an inclined plane with bounding side walls and a dam at the top of the plane to hold back a mound of liquid. The cross section of the mound is invariant in the y direction. At time $t = 0$, the dam is removed, allowing the liquid to flow down the plane. In the numerical simulation, the mound has an initial parabolic cross section of area a with a height-to-width ratio of 0.08, and occupies the top quarter of the computational plane. Bilateral symmetry is imposed on the left edge of the domain; thus it represents the centerline of the physical plane. A no-slip condition is imposed on the right edge by prohibiting downhill flow in the column of cells immediately adjacent to this edge. The leading edge of the mound at the no-slip wall continues to move, however, as a result of side flow from the interior. The speed at the wall is about one-half the average speed of the flowing front, in substantial agreement with the experimental pictures.

In Fig. 2, we show perspective views of the fluid flow down a vertical wall at two different times for $B^{-1} = 0.1$. Taking $a^{1/2}$ as characteristic length L, the dimensionless times for Figs. 2(a) and 2(b) are 16 and 120, respectively. It is seen that fingering is initiated by the no-slip condition on the right wall with the disturbance propagating inward, ultimately producing a set of relatively uniformly spaced fingers. The longest of these fingers is markedly wedge shaped, in agreement with experimental results for liquids that strongly wet the surface. Figure 3 shows the pattern generated for the same initial condition, but with $B^{-1} = 1$. The number of fingers formed (across the full channel) is seen to be reduced from five to three as a result of increasing the surface tension, say, by a factor of 10. Note the local maxima in fluid depth near the tips of the developed fingers. Also discernible in Fig. 3 is a small secondary maximum similar to that seen for the isolated drop. An explanation for these undulations, based on a simplified model, will be given below.

In the absence of the no-slip condition, fingering can also result from the imposition of a small perturbation on the uniformly propagating two-dimensional flow down the plane. Figure 4 shows the result of a small (1%) sinusoidal disturbance imposed at $t = 12$ on a slope inclined at $\alpha = 1$

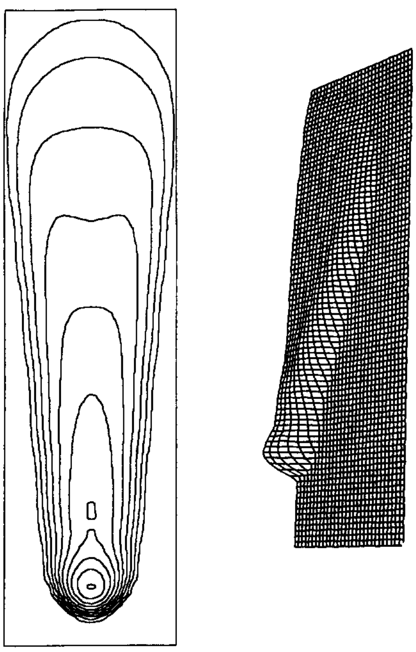

FIG. 1. Contour plot and perspective view of a droplet flowing down a vertical wall. See the text for parameter values.

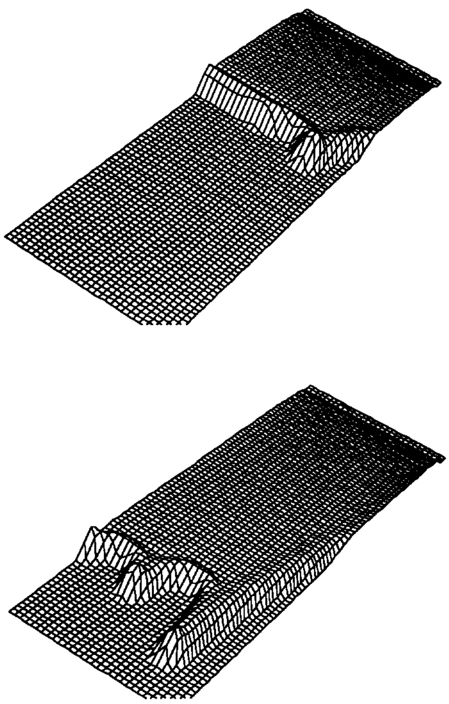

FIG. 2. Finger formation in a liquid film flowing down a vertical wall. The right edge is a no-slip side wall. Dimensionless times are $t = 16$ (upper) and $t = 120$ (lower). The fluid depth is greatly exaggerated. Surface tension parameter, $B^{-1} = 0.1$.

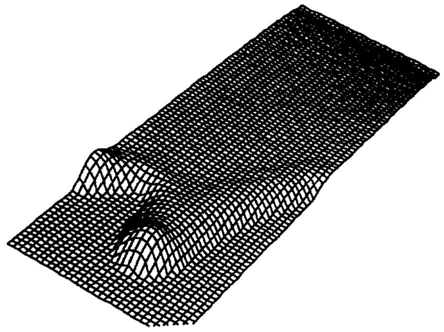

FIG. 3. As in Fig. 2, except $B^{-1} = 1.0$; $t = 120$.

rad for $B^{-1} = 0.25$. The initial mound shape is identical to that of Figs. 2 and 3. The disturbance first decays until the curvatures become sufficiently large at the front. Then lateral flow, caused by greater curvature, hence higher pressures resulting from surface tension, at the troughs, leads eventually to steady growth of the fingers. Indeed, runs were made for various α, including $\pi/2$, with $B^{-1} = 0$, both with and without the no-slip condition imposed; in none of these cases did fingering result. This confirms the prediction of Huppert[2] that surface tension is the destabilizing force.

As for the most unstable wavelength λ^* of a falling film without confining side walls, some progress can be made purely on dimensional considerations. Assuming that λ^* depends only on the initial cross-sectional area a, rather than the details of its shape (as seems to be the case in our numerical experiments), the full problem consists of Eq. (1) and the integral initial condition

$$\int_0^{x_N} h(x,t=0)dx = a, \tag{3}$$

where x_N is the downhill leading edge of the starting mound. All parameters, save one, can be absorbed in the affine scaling

$$h = a^{1/2}[\sigma/(\rho g a \sin\alpha)]^{-1/4}H, \tag{4a}$$

$$(x,y) = a^{1/2}[\sigma/(\rho g a \sin\alpha)]^{1/4}(X,Y), \tag{4b}$$

and

$$t = 3\mu a^{-5/4}\sigma^{3/4}(\rho g \sin\alpha)^{-7/4}T, \tag{4c}$$

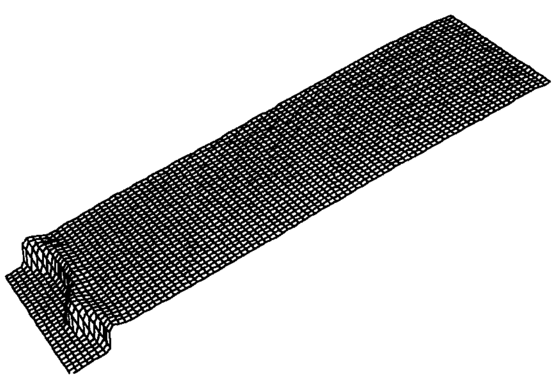

FIG. 4. The growth of a periodic imposed perturbation for a wall inclined at $\alpha = 57.3°$. The initial disturbance was an amplitude variation of 1%. Here $t = 252$; $B^{-1} = 0.25$.

where the lower-case variables in Eqs. (4) are, for clarity, original dimensional variables. Using dimensionless variables, the right side of Eq. (3) is now equal to one, while Eq. (1) becomes

$$H_T = \nabla\cdot[H^3(S\nabla H - \nabla\nabla^2 H - \mathbf{i})]. \tag{5}$$

The single remaining parameter S is equal to $[\sigma/(\rho g a \sin\alpha)]^{-1/2}\cot\alpha$. For a vertical wall $S = 0$ and the most unstable wavelength (an in-plane length) must scale as

$$\lambda^* = \text{const}\, a^{1/2}[\sigma/(\rho g a)]^{1/4}, \tag{6}$$

where, based on the simulations shown in Figs. 2 and 3, we estimate the constant of proportionality to be about 4. For finite slope, dimensional considerations predict

$$\lambda^* = a^{1/2}[\sigma/(\rho g a \sin\alpha)]^{1/4}f(S),$$

where f is some function of its argument. This does not appear to be completely consistent with the one-third power dependence predicted by Huppert[2] and validated experimentally by Silvi and Dussan V[3] as well as himself.

The origin of the undulations near the moving front is suggested by a linearized form of the governing equation that admits a damped sinusoidal solution. Equation (5) may be written for one space dimension as

$$H_T = [H^3(SH_X - H_{XXX} - 1)]_X. \tag{7}$$

We seek an approximate solution by assuming that the waveform is fixed to the front and moves with constant speed U and that the fluid depth is almost constant. Then $H = f(\xi)$, where $\xi = X - UT$ and the resulting equation may be linearized by using an expansion of the form

$$f = U^{1/2} + \epsilon e^{k\xi} + o(\epsilon).$$

Here k can be shown to satisfy a cubic equation with the relevant roots corresponding to a decaying sinusoid. For $S = 0$, k may be found explicitly as $3^{1/3}U^{-1/6}(1 + 3^{1/2}i)/2$. The damping is larger for finite S; thus the undulations may not be detectable when α is small.

We conclude that fingering on a finite slope is an inherent phenomena caused by surface tension and that, even with perfect wetting, it must ultimately occur. For slopes less than vertical, without gross perturbations such as a confining side wall, some or all disturbances will first be damped. Eventually, however, the evolving profile will develop sufficiently high curvature at the moving front that small disturbances will initiate fingering. For practical applications, involving plates of finite length, a sufficiently small inclination angle may remove this usually undesirable effect. We are currently exploring the extent to which the finite advancing contact angle may modify the above conclusions as well as pursuing a more direct stability analysis for the unconfined problem.

ACKNOWLEDGMENT

This work was supported by the National Science Foundation under Grant No. MSM-8707856.

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